Solution 1

(i) The fundamental quantity is mass.

(ii) Mass Balance Equation:

\[
\frac{d}{dt} [\text{Total Mass}] = M_{\text{in}} - M_{\text{out}}
\]

Total mass of the system, \( M_{\text{total}} = \rho V \), is the total mass of the conical tank. For a cone:

\[
V_{\text{cone}} = \frac{1}{3} \pi r^2 h
\]

For a cone, the ratio of its radius \( r \) to the height \( h \) at which the radius is measured always stays constant. So, if the radius at the base of the cone (at the top of the tank) is \( R \) and the total height is \( H \), then the following holds true:

\[
\frac{r}{h} = \frac{R}{H} \implies r = \frac{Rh}{H}
\]

Substituting,

\[
V = \frac{1}{3} \pi \frac{R^2}{H^2} h^3
\]

\[
\frac{d}{dt} (\rho V) = \rho \pi \frac{R^2}{H^2} h^2 \frac{dh}{dt}
\]

\[
M_{\text{in}} = \rho F_i, \quad M_{\text{out}} = \rho \beta \sqrt{h}
\]

\[
\implies \rho \pi \frac{R^2}{H^2} h^2 \frac{dh}{dt} = \rho F_i - \rho \beta \sqrt{h}
\]

\[
\frac{dh}{dt} = \frac{H^2}{\pi R^2} \left( \frac{F_i}{h^2} - \frac{\beta}{h^2} \right)
\]

(iii) Consistency:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variables of the model</td>
<td>5</td>
</tr>
<tr>
<td>Constants</td>
<td>3</td>
</tr>
<tr>
<td>Inputs and Disturbances</td>
<td>1</td>
</tr>
<tr>
<td>Outputs (Unknowns)</td>
<td>1</td>
</tr>
<tr>
<td># of Independent Equations</td>
<td>1</td>
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</tbody>
</table>
The model is consistent because the total number of model variables minus all constants, inputs and disturbances is equal to the number of outputs and independent equations.

(iv) Only $F_i$ can be treated as an input. It could be either a manipulated variable or a disturbance. A pump or a valve will be required for it to be manipulated.

(v) The level $h$ is the controlled variable. It may or may not be measured.

(vi) The process model is nonlinear due to the square root dependence of the outlet flow and the shape of the tank.

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**Solution 2**

This is an example of a feedback control scheme. The outlet temperature is controlled by manipulating the flow of the inlet stem equipped with a pump. The liquid level in the tank is controlled by manipulating the outlet flow.

(i) In this problem we identify:

- Control variables: $T, L$ (assumed to be measured)
- Manipulated variables: $F_2, F_3$
- Disturbances: $F_1, T_1, T_2$ (which can be either measured or unmeasured)

(ii) Fundamental quantities:

- Mass is required to describe variations in the level $L$.
- Energy is required to describe variations in the temperature $T$.

(iii) Mass Balance:

\[
\text{Total Mass} = \rho V = \rho AL
\]

where $A$ is the cross-sectional area of the tank. Balance:

\[
\frac{d(\rho AL)}{dt} = \rho F_1 + \rho F_2 - \rho F_3 \Rightarrow \frac{dL}{dt} = \frac{1}{A} (F_1 + F_2 - F_3)
\]

Energy balance: The energy balance can be simplified to an enthalpy balance (assuming no heat loss/gain to/from the tank):

\[
\text{Total Enthalpy} = \rho C_p V (T - T_{ref})
\]
Therefore:
\[
\frac{d}{dt} \left( \rho C_p V \left( T - T_{ref} \right) \right) = \frac{d}{dt} \left( \rho C_p A L \left( T - T_{ref} \right) \right) = \rho C_p A \frac{d}{dt} \left( L \left( T - T_{ref} \right) \right)
\]

Because both \( L \) and \( T \) are changing, we have to use the product rule to simplify this expression further. This yields:
\[
\rho C_p A \frac{d}{dt} \left( L \left( T - T_{ref} \right) \right) = \rho C_p A \left[ L \frac{d}{dt} \left( T - T_{ref} \right) + \left( T - T_{ref} \right) \frac{dL}{dt} \right]
\]

Note that \( T_{ref} \) is a constant. Therefore:
\[
\frac{d}{dt} \left( T - T_{ref} \right) = \frac{dT}{dt}
\]

Substituting in our expression for \( \frac{dT}{dt} \), we produce an expression for the accumulation of enthalpy in our system:
\[
\frac{d}{dt} \left( E_{sys} \right) = \rho C_p A L \frac{dT}{dt} + \rho C_p A \left( T - T_{ref} \right) \left[ \frac{1}{A} (F_1 + F_2 - F_3) \right]
\]

For the right hand side of the overall enthalpy balance, we have:

Enthalpy in \( = \rho C_p F_1 (T_1 - T_{ref}) + \rho C_p F_2 (T_2 - T_{ref}) \)

Enthalpy out \( = \rho C_p F_3 (T - T_{ref}) \)

Combining equations to finalize the balance produces:

\[
\rho C_p \left( A L \frac{dT}{dt} + (T - T_{ref}) (F_1 + F_2 - F_3) \right) = \rho C_p (F_1 (T_1 - T_{ref}) + F_2 (T_2 - T_{ref}) - F_3 (T - T_{ref}))
\]

All the \( T_{ref} \) terms cancel out, along with \( \rho C_p \):
\[
AL \frac{dT}{dt} + (TF_1 + TF_2 - TF_3) = F_1 T_1 + F_2 T_2 - F_3 T
\]

The \( F_3 T \) terms cancel each other out as well, leaving us with our final expression:
\[
\frac{dT}{dt} = \frac{F_1}{AL} (T_1 - T) + \frac{F_2}{AL} (T_2 - T)
\]

Thus, our final model that describes the behavior of our outputs (controlled variables) is given by:
\[
\frac{dL}{dt} = \frac{1}{A} (F_1 + F_2 - F_3)
\]
\[
\frac{dT}{dt} = \frac{F_1}{AL} (T_1 - T) + \frac{F_2}{AL} (T_2 - T)
\]
(iv) Consistency:

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<tr>
<td>Model Variables</td>
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<tr>
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The model is consistent because the total number of model variables minus all constants, inputs and disturbances is equal to the number of outputs and independent equations.

(v) The model is nonlinear due to the presence of the term $-\frac{F₁T}{AL}$ that appears in the equation for $\frac{dT}{dt}$. A model becomes nonlinear when outputs or inputs are either:

- a) raised to a power and/or,
- b) multiplied or divided by each other.

That is to say, a nonlinear system is any system in which the unknowns cannot be expressed as a linear combination of the model’s inputs and unknowns.