Transfer Function Models
of Dynamical Processes

Process Dynamics and Control
Linear SISO Control Systems

- General form of a linear SISO control system:

\[
\frac{d^n y}{dt} + a_n \frac{d^{(n-1)} y}{dt^{(n-1)}} + \ldots + a_0 y = \\
\quad b_n \frac{d^{(n-1)} u}{dt^{(n-1)}} + \ldots + b_1 \frac{du}{dt} + b_0 u
\]

- this is a underdetermined higher order differential equation

- the function \( u(t) \) must be specified for this ODE to admit a well defined solution
Heated stirred-tank model (constant flow, \( F \))

\[
\frac{dT}{dt} = \frac{F}{V} (T_{in} - T) + \frac{Q}{\rho C_p V}
\]

Taking the Laplace transform yields:

\[
sT(s) - T(0) = \frac{F}{V} T_{in}(s) - \frac{F}{V} T(s) + \frac{1}{\rho C_p V} Q(s)
\]

or letting \( \tau = \frac{V}{F} \)

\[
T(s) = \frac{1}{\tau s + 1} T(0) - \frac{1}{\tau s + 1} T_{in}(s) + \frac{1}{\tau s + 1} \frac{1}{\rho C_p F} Q(s)
\]
Heated stirred tank example

\[ T(s) = \frac{1}{\tau s + 1}T(0) + \frac{1}{\tau s + 1}T_{in}(s) + \frac{1}{\tau s + 1}\frac{\rho C_p l^2}{T(s)} \]

e.g. The block \( \frac{K}{\tau s + 1} \) is called the transfer function relating \( Q(s) \) to \( T(s) \)
Process Control

Ability to understand dynamics in Laplace and time domains is extremely important in the study of process control.
Transfer functions

- Transfer functions are generally expressed as a ratio of polynomials

\[ G(s) = \frac{B(s)}{A(s)} \]

Where

\[ A(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \]

\[ B(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1s + b_0 \]

- The polynomial \( A(s) \) is called the characteristic polynomial of \( G(s) \)

- Roots of \( B(s) = 0 \) are the zeroes of \( G(s) \)

- Roots of \( A(s) = 0 \) are the poles of \( G(s) \)
Order of underlying ODE is given by degree of characteristic polynomial

e.g. First order processes

\[ Y(s) = \frac{K}{\tau s + 1} U(s) \]

Second order processes

\[ Y(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} U(s) \]

\[ Y(s) = \frac{a_1 s + a_0}{\tau^2 s^2 + 2\xi \tau s + 1} U(s) \]

*Order* of the process is the degree of the characteristic (denominator) polynomial

*The relative order* is the difference between the degree of the denominator polynomial and the degree of the numerator polynomial
Steady state behavior of the process obtained from the final value theorem
e.g. First order process

\[ Y(s) = \frac{K}{\tau s + 1} U(s) \]

For a unit-step input,

\[ Y(s) = \frac{K}{s(\tau s + 1)} \]

From the final value theorem, the ultimate value of \( y(t) \) is

\[ \lim_{t \to \infty} y(t) = \lim_{s \to 0} \left[ sY(s) \right] = K \]

- This implies that the limit exists, i.e. that the system is stable.
Transfer function

Transfer function is the unit impulse response

e.g. First order process,

\[ Y(s) = \frac{K}{\tau s + 1} U(s) \]

Unit impulse response is given by

\[ Y(s) = \frac{K}{\tau s + 1} \]

In the time domain,

\[ y(t) = \frac{K}{\tau} e^{-\frac{1}{\tau} t} \]
Transfer Function

- Unit impulse response of a 1st order process

\[ y(t) = \frac{K}{\tau} e^{-\frac{1}{\tau} t} \]

\[ K = 1, \quad \tau = 1 \]
Deviation Variables

- To remove dependence on initial condition

\[ \frac{dT}{dt} = \frac{F}{V} (T_{in} - T) + \frac{Q}{\rho C_p V} \]

Compute equilibrium condition for a given \( Q = Q_s \) and \( T_{in} = T_{ins} \)

\[ 0 = \frac{F}{V} (T_{ins} - T_s') + \frac{Q_s}{\rho C_p V} \]

Define deviation variables

\[ T' = T - T_s, \quad Q' = Q - Q_s, \quad T'_{in} = T_{in} - T_{ins} \]

Rewrite linear ODE

\[ \frac{dT'}{dt} = \frac{F}{V} (T'_{in} - T') + \frac{Q'}{\rho C_p V} + \frac{F}{V} (T_{ins} - T_s) + \frac{Q_s}{\rho C_p V} \]

or

\[ \frac{dT'}{dt} = \frac{F}{V} (T'_{in} - T') + \frac{Q'}{\rho C_p V} \]
Deviation Variables

Assume that we start at equilibrium \( T'(0) = 0 \)

\[
T'(s) = \frac{1}{\tau s + 1} T'_{\text{in}} + \frac{K}{\tau s + 1} Q'(s)
\]

*Transfer functions express extent of deviation from a given steady-state*

- **Procedure**
  - Find steady-state
  - Write steady-state equation
  - Subtract from linear ODE
  - Define deviation variables and their derivatives if required
  - Substitute to re-express ODE in terms of deviation variables
Process Modeling

Gravity tank

Objectives: height of liquid in tank
Fundamental quantity: Mass, momentum
Assumptions:
- Outlet flow is driven by head of liquid in the tank
- Incompressible flow
- Plug flow in outlet pipe
- Turbulent flow
From mass balance and Newton’s law,

\[
\frac{dh}{dt} = \frac{F_o}{A} - \frac{A_p v}{A}
\]

\[
\frac{dv}{dt} = \frac{h g}{L} - \frac{K_f v^2}{\rho A_p L}
\]

A system of simultaneous ordinary differential equations results

Linear or nonlinear?
Q: If the model of the process is nonlinear, how do we express it in terms of a transfer function?

A: We have to approximate it by a linear one (i.e. Linearize) in order to take the Laplace.
Nonlinear systems

- First order Taylor series expansion

1. Function of one variable

\[ f(x) \approx f(x_s) + \frac{\partial f(x_s)}{\partial x}(x - x_s) \]

2. Function of two variables

\[ f(x, u) \approx f(x_s, u_s) + \frac{\partial f(x_s, u_s)}{\partial x}(x - x_s) + \frac{\partial f(x_s, u_s)}{\partial u}(u - u_s) \]

3. ODEs

\[ \dot{x} = f(x) \approx f(x_s) + \frac{\partial f(x_s)}{\partial x}(x - x_s) \]
Procedure to obtain transfer function from nonlinear process models

- Find an equilibrium point of the system
- Linearize about the steady-state
- Express in terms of deviations variables about the steady-state
- Take Laplace transform
- Isolate outputs in Laplace domain

\[
Y(s) = G_1(s)U_1(s) + G_2(s)U_2(s)
\]

- Express effect of inputs in terms of transfer functions

\[
G_1(s) = \left. \frac{Y(s)}{U_1(s)} \right|_{U_2(s)=0}
\]

\[
G_2(s) = \left. \frac{Y(s)}{U_2(s)} \right|_{U_1(s)=0}
\]
Block Diagrams

- Transfer functions of complex systems can be represented in block diagram form.

- 3 basic arrangements of transfer functions:
  1. Transfer functions in series
  2. Transfer functions in parallel
  3. Transfer functions in feedback form
Transfer functions in series

Overall operation is the multiplication of transfer functions

\[ Y_2(s) = G_2(s)G_1(s)U(s) \]

Resulting overall transfer function
Transfer functions in series (two first order systems)

Overall operation is the multiplication of transfer functions

\[
Y_2(s) = \frac{K_2}{\tau_2 s + 1} \frac{K_1}{\tau_1 s + 1} U(s)
\]

Resulting overall transfer function
Transfer Functions

- **DC Motor example:**
  - In terms of angular velocity
  \[
g_1(s) = \frac{\Omega(s)}{V_a(s)} = \frac{K}{\tau s + 1}
\]
  - In terms of the angle
  \[
g_2(s) = \frac{\Theta(s)}{\Omega(s)} = \frac{1}{s}
\]

\[
G(s) = \frac{\Theta(s)}{V_a(s)} = \frac{K}{s(\tau s + 1)} = g_2(s)g_1(s)
\]
Transfer Functions

- Transfer function in parallel

\[ Y(s) = (G_1(s) + G_2(s))U(s) \]
Transfer Functions

- Transfer function in parallel

\[ \frac{K_1}{\tau_1 s + 1} \quad \frac{K_2}{\tau_2 s + 1} \]

Overall transfer function is the addition of TFs in parallel

\[ Y(s) = \left( \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2 s + 1} \right) U(s) \]
Transfer Functions

- Transfer functions in (negative) feedback form

\[ Y(s) = G_1(s)(U(s) - G_2(s)Y(s)) \]
\[ (1 + G_1(s)G_2(s))Y(s) = G_1(s)U(s) \]

\[ Y(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)} U(s) \]
Transfer Functions

- Transfer functions in (positive) feedback form

\[ Y(s) = G_1(s)(U(s) + G_2(s)Y(s)) \quad \rightarrow \quad (1 - G_1(s)G_2(s))Y(s) = G_1(s)U(s) \]

\[ Y(s) = \frac{G_1(s)}{1-G_1(s)G_2(s)} U(s) \]
Transfer Function

Example 3.20

\[
\frac{Y(s)}{R(s)} = \frac{G_1G_2G_5 + G_1G_6}{1 - G_1G_3 + G_1G_2G_4}
\]
Transfer Function

Example

\[ G_7 = \frac{G_1}{1 - G_1 G_3} \]

- A positive feedback loop
Example 3.20

Two systems in parallel

Replace $G_6$ by $G_2 \left( \frac{G_6}{G_2} \right)$
Example 3.20

Two systems in parallel
Transfer Function

Example 3.20

A negative feedback loop

\[
\frac{G_1}{1-G_1G_3} G_2 \quad \frac{G_5 + \frac{G_6}{G_2}}{1+\left(\frac{G_1}{1-G_1G_3}\right) G_2 G_4}
\]
Transfer Function

Example 3.20

Two processes in series

\[
\frac{Y(s)}{R(s)} = \left( \frac{G_1}{1 - G_1 G_3} \right) \frac{G_2}{1 + \left( \frac{G_1}{1 - G_1 G_3} \right) G_2 G_4} \left( G_5 + \frac{G_6}{G_2} \right)
\]

\[
\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_5 + G_1 G_6}{1 - G_1 G_3 + G_1 G_2 G_4}
\]
First Order Systems

First order systems are systems whose dynamics are described by the transfer function

\[ G(s) = \frac{K}{\tau s + 1} \]

where

- \( K = G(0) \) is the system’s (steady-state) gain
- \( \tau \) is the time constant

First order systems are the most common behaviour encountered in practice
First Order Systems

Examples, Liquid storage

Assume:

- Incompressible flow
- Outlet flow due to gravity \( F \propto h \)

Balance equation:

- Total \( \rho V = \rho Ah \)
- Flow In \( \rho F_{in} \)
- Flow Out \( \rho F = \beta h \)
First Order Systems

**Balance equation:**

\[ \rho A \frac{dh}{dt} = \rho F_{in} - \beta h \]

- Deviation variables about the equilibrium \( h(0) = 0, \ F_{in} = 0 \)

\[ \rho A \frac{dh'}{dt} = \rho F'_{in} - \beta h' \]

- Laplace transform

\[ \rho A s H(s) = \rho F_{in}(s) - \beta H(s) \]

\[ (\rho A s + \beta) H(s) = \rho F_{in}(s) \]

\[ G(s) = \frac{H(s)}{F_{in}(s)} = \frac{\rho}{\rho A s + \beta} \times \frac{1/\beta}{1/\beta} = \frac{\rho / \beta}{\beta s + 1} \]

**First order system with**

\[ K = \rho / \beta, \ \tau = \rho A / \beta \]
First Order Systems

Examples: Cruise control

\[ \frac{V(s)}{U(s)} = \frac{1}{m_{\text{car}}} \times \frac{m_{\text{car}}/b}{b/m_{\text{car}} + 1} \]

\[ K = \frac{1}{b}, \quad \tau = \frac{b}{m_{\text{car}}} \]

\[ G(s) = \frac{K}{\tau s + 1} \]

DC Motor

\[ \frac{\Omega(s)}{V_a(s)} = \frac{K_t}{J_m R_a} \left( s + \left( \frac{b}{J_m} + \frac{K_e K_t}{J_m R_a} \right) \right) \]

\[ K = \frac{K_t}{b R_a + K_e K_t} \]

\[ \tau = \frac{1}{b R_a + K_e K_t} \]

\[ G(s) = \frac{K}{\tau s + 1} \]
First Order Systems

<table>
<thead>
<tr>
<th></th>
<th>$K$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Liquid Storage Tank</strong></td>
<td>$K = \rho/\beta$</td>
<td>$\tau = \rho A/\beta$</td>
</tr>
<tr>
<td><strong>Speed of a car</strong></td>
<td>$K = 1/b$</td>
<td>$\tau = b/m_{\text{car}}$</td>
</tr>
<tr>
<td><strong>DC Motor</strong></td>
<td>$K = \frac{K_t}{bR_a + K_e K_t}$</td>
<td>$\tau = \frac{1}{bR_a + K_e K_t}$</td>
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First order processes are characterized by:

1. Their capacity to store material, momentum and energy
2. The resistance associated with the flow of mass, momentum or energy in reaching their capacity
First Order Systems

- **Step response of first order process**

\[ Y(s) = \frac{KM}{s(\tau s + 1)} \]

Step input signal of magnitude M

\[ y(t) = KM \left( 1 - \tau e^{-\frac{t}{\tau}} \right) \]

- The ultimate change in \( y(t) \) is given by

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[ KM \left( 1 - \tau e^{-\frac{t}{\tau}} \right) \right] = KM \]
First Order Systems

- Step response
First Order Systems

- **What do we look for?**
  - **System’s Gain: Steady-State Response**
    \[
    \lim_{s \to 0} \left[ \frac{K}{\tau s + 1} \right] = K = \frac{\text{Overall Change in } y}{\text{Overall Change in } u}
    \]
  - **Process Time Constant:**
    \[
    \tau = \text{Time Required to Reach 63.2% of final value}
    \]

- **What do we need?**
  - **System initially at equilibrium**
  - **Step input of magnitude } M**
  - **Measure process gain from new steady-state**
  - **Measure time constant**
First Order Systems

- First order systems are also called systems with finite settling time

  - The settling time $t_s$ is the time required for the system comes within 5% of the total change and stays 5% for all times $t > t_s$

  - Consider the step response

    $\frac{y(t)}{KM} = \left( 1 - e^{-\frac{t}{\tau}} \right)$

  - The overall change is

    - $t = \tau$ \hspace{0.5cm} $\frac{y(\tau)}{KM} = 0.632$

    - $t = 2\tau$ \hspace{0.5cm} $\frac{y(2\tau)}{KM} = 0.865$

    - $t = 3\tau$ \hspace{0.5cm} $\frac{y(3\tau)}{KM} = 0.950$

    $t_s \approx 3\tau$
First Order Systems

- Settling time

![Graph showing step response with settling time and 5% criterion]
First Order Systems

Process initially at equilibrium subject to a step of magnitude 1

\[ K = 3.2, \tau = 7 \]
First order process

Ramp response:

\[ Y(s) = \frac{K}{\tau s + 1} \frac{a}{s^2} \quad \rightarrow \quad y(t) = Ka \left( t - \tau \left(1 - e^{-\frac{t}{\tau}}\right) \right) \]

Ramp input of slope a

```
+-------------------+---+
|                  t |
+-------------------+---+
| 0  0.5  1  1.5  2 |
+-------------------+---+
| 0   0.5  1  1.5  2 |
+-------------------+---+
| 0   0.5  1  1.5  2 |
+-------------------+---+
| 0  0.5  1  1.5  2 |
+-------------------+---+
| 0  0.5  1  1.5  2 |
+-------------------+---+
| 0  0.5  1  1.5  2 |
+-------------------+---+
```

\[ \frac{y(t)}{K} \]

\[ t \]

\[ a \]

\[ \tau \]
First Order Systems

Sinusoidal response \( u(t) = A \sin \omega t \)

\[
Y(s) = \frac{K}{\tau s + 1} \frac{A \omega}{s^2 + \omega} \quad \Rightarrow \quad Y(s) = \frac{KA}{1 + \omega^2 \tau^2} \left( \sin(\omega t) + \tau \omega \left( e^{-\frac{t}{\tau}} - \cos(\omega t) \right) \right)
\]

\[
\lim_{t \to \infty} [y(t)] = \frac{KA}{\sqrt{1 + \tau^2 \omega^2}} \sin(\omega t + \phi)
\]
First Order Systems

Amplitude Ratio

\[ AR = \frac{K}{\sqrt{1 + \tau^2 \omega^2}} \]

Phase Shift

\[ \phi = -\arctan \omega \tau \]
Example: Liquid storage tank

Laplace domain dynamics

\[ H(s) = \frac{1}{As} F_{in}(s) - \frac{1}{As} F(s) \]

If there is no outlet flow, \( F = 0 \)

\[ H(s) = \frac{1}{As} F_{in}(s) \]
Example

- Capacitor

\[
\begin{align*}
\text{Dynamics of both systems is equivalent}
\end{align*}
\]
Integrating Systems

- Step input of magnitude M

\[ Y(s) = \frac{K}{s} \frac{M}{s} = \frac{KM}{s^2} \]

\[ y(t) = \begin{cases} 
0 & t < 0 \\
KMt & t \geq 0
\end{cases} \]
Integrating Systems

- Unit impulse response

\[ Y(s) = \frac{K}{s} M = \frac{KM}{s} \]

\[ y(t) = \begin{cases} 
0 & t < 0 \\
KM & t \geq 0 
\end{cases} \]
Integrating Systems

- Rectangular pulse response

\[ Y(s) = \frac{K}{s} \frac{M}{s} (1 - e^{-t_w s}) = \frac{KM}{s^2} (1 - e^{-t_w s}) \]

\[ y(t) = \begin{cases} 
0 & t < 0 \\
KMt & t < t_w \\
KMt_w & t \geq t_w 
\end{cases} \]
Second order Systems

- Second order process:
  - Assume the general form

\[ Y(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} U(s) \]

where

- \( K \) = Process steady-state gain
- \( \tau \) = Process time constant
- \( \xi \) = Damping Coefficient

- Three families of processes

\[ \xi < 1 \quad \text{Underdamped} \\
\xi = 1 \quad \text{Critically Damped} \\
\xi > 1 \quad \text{Overdamped} \]
Second Order Systems

Three types of second order process:

1. Two First Order Systems in series or in parallel
e.g. Two holding tanks in series

2. Inherently second order processes: Mechanical systems possessing inertia and subjected to some external force
   e.g. A pneumatic valve

3. Processing system with a controller: Presence of a controller induces oscillatory behavior
   e.g. Feedback control system
Second order Systems

- Multicapacity Second Order Processes
  - Naturally arise from two first order processes in series

\[ U(s) \xrightarrow{\frac{K_1}{\tau_1 s+1}} \frac{K_2}{\tau_2 s+1} \xrightarrow{} Y(s) \]

\[ U(s) \xrightarrow{\frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)}} \xrightarrow{} Y(s) \]

- By multiplicative property of transfer functions

\[ \frac{Y(s)}{U(s)} = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)} \]
First order systems in parallel

Overall transfer function a second order process (with one zero)

\[
Y(s) = \left( \frac{K_1 \tau_2 + K_2 \tau_1}{(\tau_1 s + 1)(\tau_2 s + 1)} + \frac{K_1 + K_2}{\tau_1 s + 1} \right) U(s)
\]
Inherently second order process:
e.g. Pneumatic Valve

By Newton’s law

\[ M \ddot{x} = pA - Kx - C \dot{x} \]

\[ Ms^2 X(s) = AP(s) - KX(s) - C sX(s) \]

\[ G(s) = \frac{X(s)}{P(s)} = \frac{A/K}{M/K s^2 + C/K s + 1} \]
Feedback Control Systems

\[
G(s) = \frac{\frac{K_c \tau_i s + K_c}{\tau_i s}}{1 + \frac{K_c \tau_i s + K_c}{\tau_i s} \frac{K}{\tau s + 1}} = \frac{K_c K \tau_i s + K_c K}{\tau_i \tau s^2 + (\tau_i + K_c K \tau_i) s + K_c K}
\]
Second order Systems

- Second order process:
  - Assume the general form

\[ Y(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} U(s) \]

where
- \( K \) = Process steady-state gain
- \( \tau \) = Process time constant
- \( \xi \) = Damping Coefficient

- Three families of processes

\( \xi < 1 \) Underdamped
\( \xi = 1 \) Critically Damped
\( \xi > 1 \) Overdamped
Second Order Systems

- Roots of the characteristic polynomial

\[-\frac{\xi}{\tau} \pm \frac{1}{\tau} \sqrt{\xi^2 - 1}\]

Case 1) $\xi > 1$  Two distinct real roots
System has an exponential behavior

Case 2) $\xi = 1$  One multiple real root
Exponential behavior

Case 3) $\xi < 1$  Two complex roots
System has an oscillatory behavior
Second Order Systems

Step response of magnitude $M$

$$Y(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} \frac{M}{s}$$
Second Order Systems

Observations

- Responses exhibit overshoot \((y(t)/KM > 1)\) when \(\xi < 1\)
- Large \(\xi\) yield a slow sluggish response
- Systems with \(\xi = 1\) yield the fastest response without overshoot
- As \(\xi\) (with \(\xi < 1\)) becomes smaller, system becomes more oscillatory
Second Order Systems

Characteristics of underdamped second order process

1. Rise time, $t_r$
2. Time to first peak, $t_p$
3. Settling time, $t_s$
4. Overshoot:

$$OS = \frac{b}{a} = \exp\left(-\frac{\xi}{\sqrt{1-\xi^2}}\pi\right)$$

5. Decay ratio:

$$DR = \frac{c}{b} = \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\right)$$
Second Order Systems

- Step response
Second Order Systems

- Sinusoidal Response

\[
Y(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} \frac{A\omega}{s^2 + \omega^2}
\]

\[
y_\infty(t) = \frac{KA}{\sqrt{(1-\omega^2\tau^2)^2 + 4\xi^2\tau^2}} \sin(\omega t + \phi)
\]

where \(\phi = -\arctan\left(\frac{2\xi\tau\omega}{1-\omega^2\tau^2}\right)\) and \(AR = \frac{K}{\sqrt{(1-\omega^2\tau^2)^2 + 4\xi^2\tau^2}}\)
Second Order Systems

![Bode Diagram](image)

- **Magnitude (dB)**
- **Phase (deg)**

- **Frequency (rad/sec)**

- **Graphs for different values of** $\xi$:
  - $\xi = 0.1$
  - $\xi = 0.5$
  - $\xi = 1$
  - $\xi = 2$
More Complicated Systems

Transfer function typically written as rational function of polynomials

\[ G(s) = \frac{B(s)}{A(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \]

where \( B(s) \) and \( A(s) \) can be factored following the discussion on partial fraction expansion

\[ A(s) = (s - p_1)(s - p_2) \cdots (s - p_n) \]
\[ B(s) = (s - z_1)(s - z_2) \cdots (s - z_{n-1}) \]

s.t.

\[ G(s) = \frac{(s-z_1)(s-z_2)\ldots(s-z_{n-1})}{(s-p_1)(s-p_2)\ldots(s-p_n)} \]

where \( z_i \) and \( p_j \) appear as real numbers or complex conjugate pairs
Poles and zeroes

Definitions:

- the roots of $B(s)$ are called the **zeros** of $G(s)$

\[ s = z_1, \ s = z_2, \ldots, \ s = z_{n-1} \]

- the roots of $A(s)$ are called the **poles** of $G(s)$

\[ s = p_1, \ s = p_2, \ldots, \ s = p_n \]

Poles: Directly related to the underlying differential equation

If $\Re(p_i) < 0$, then there are terms of the form $e^{\Re(p_i)t}$ in $y(t)$

\[ y(t) \text{ vanishes to a unique point} \]

If any $\Re(p_i) > 0$ then there is at least one term of the form $e^{\Re(p_i)t}$

\[ y(t) \text{ does not vanish} \]
e.g. A transfer function of the form

$$\frac{K}{s(\tau_1 s + 1)(\tau_2^2 s^2 + 2\xi \tau_2 s + 1)}$$

with $0 \leq \xi < 1$ can factored to a sum of

- A constant term from $s$
- $e^{-t/\tau_1}$ from the term $(\tau_1 s + 1)$
- A function that includes terms of the form

$$e^{-\xi t/\tau_2} \sin \left( \frac{t\sqrt{1-\xi^2}}{\tau_2} \right)$$

$$e^{-\xi t/\tau_2} \cos \left( \frac{t\sqrt{1-\xi^2}}{\tau_2} \right)$$

Poles can help us to describe the qualitative behavior of a complex system (degree>2)
The sign of the poles gives an idea of the stability of the system
Poles

- Calculation performed easily in MATLAB
- Function POLE, PZMAP

\[ G(s) = \frac{1}{s(s+1)(4s^2+2s+1)} \]

```
» s=tf('s');
» sys=1/s/(s+1)/(4*s^2+2*s+1);
Transfer function:
    1
-------------------------
4 s^4 + 6 s^3 + 3 s^2 + s
-------------------------
```

```matlab
» pole(sys)
ans =
    0
   -1.0000
-0.2500 + 0.4330i
-0.2500 - 0.4330i
```
Poles

- Function PZMAP
  
  ```matlab
  >> pzmap(sys)
  ```

**MATLAB**

![Pole-Zero Map](image)
Poles

- One constant pole
  - integrating feature

- One real negative pole
  - decaying exponential

- A pair of complex roots with negative real part
  - decaying sinusoidal

What is the dominant feature?
Poles

- **Step Response**

![Step Response Graph]

- Integrating factor dominates the dynamic behaviour
Example

\[ G(s) = \frac{1}{s^3 + s^2 + s + 1} \]

- Poles
  - -1.0000
  - -0.0000 + 1.0000i
  - -0.0000 - 1.0000i

- One negative real pole
- Two purely complex poles

What is the dominant feature?
Purely complex poles dominate the dynamics
Example

\[ G(s) = \frac{1}{s^3 + 9s^2 + 7s + 1} \]

- Three negative real poles

What is the dominant dynamic feature?
Poles

- **Step Response**

- Slowest exponential ($p_1 = -0.1868$) dominates the dynamic feature (yields a time constant of $1/0.1868 = 5.35$ s)
Example

\[ G(s) = \frac{1}{s^3 + 4s^2 - 3s + 1} \]

- Poles
  - -4.6858
  - 0.3429 + 0.3096i
  - 0.3429 - 0.3096i

- One negative real root
- Complex conjugate roots with \textit{positive real} part

- The system is unstable (grows without bound)
Poles

- Step Response

![Step Response Graph]

- Amplitude vs Time (sec)
Two types of poles

- Slow (or dominant) poles
  - Poles that are closer to the imaginary axis
    - Smaller real part means smaller exponential term and slower decay

- Fast poles
  - Poles that are further from the imaginary axis
    - Larger real part means larger exponential term and faster decay
Poles dictate the stability of the process

- Stable poles
  - Poles with negative real parts
  - Decaying exponential
- Unstable poles
  - Poles with positive real parts
  - Increasing exponential
- Marginally stable poles
  - Purely complex poles
  - Pure sinusoidal
Poles

Poles

Stable

Unstable

Marginally Stable

Pure Exponential

Oscillatory Behaviour

Integrating Behaviour

Re

Im
Poles

Example

- Stable Poles
- Unstable Poles
- Fast Poles
- Slow Poles

Pole-Zero Map

Real Axis

Imaginary Axis
Poles and zeroes

Definitions:

- The roots of $B(s)$ are called the **zeros** of $G(s)$

\[ s = z_1, \ s = z_2, \ldots, \ s = z_{n-1} \]

- The roots of $A(s)$ are called the **poles** of $G(s)$

\[ s = p_1, \ s = p_2, \ldots, \ s = p_n \]

Zeros:

- Do not affect the stability of the system but can modify the dynamical behaviour
Zeros

- Types of zeros:

  - **Slow zeros** are closer to the imaginary axis than the dominant poles of the transfer function
    - Affect the behaviour
    - Results in overshoot (or undershoot)
  
  - **Fast zeros** are further away to the imaginary axis
    - have a negligible impact on dynamics

  - Zeros with negative real parts, $\text{Re}(z_i) < 0$, are **stable zeros**
    - Slow stable zeros lead to overshoot

  - Zeros with positive real parts, $\text{Re}(z_i) > 0$, are **unstable zeros**
    - Slow unstable zeros lead to undershoot (or inverse response)
Zeros

Can result from two processes in parallel

\[ G(s) = K \frac{\tau_a s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)} \]

\[ K = K_1 + K_2 \]

\[ \tau_a = \frac{K_1 \tau_2 + K_2 \tau_1}{K_1 + K_2} \]

If gains are of opposite signs and time constants are different then a right half plane zero occurs
Zeros

Example

\[ G(s) = \frac{\frac{1}{10} s+1}{s^3+9s^2+7s+1} \]

- Poles
  - -8.1569
  - -0.6564
  - -0.1868

- Zeros
  - -10.000

*What is the effect of the zero on the dynamic behaviour?*
Zeros

- Poles and zeros

![Pole-Zero Map](image)

- Fast, Stable Zero
- Dominant Pole
Zeros

- Unit step response:

- Effect of zero is negligible
Example

\[ G(s) = \frac{10s+1}{s^3+9s^2+7s+1} \]

- **Poles**
  - -8.1569
  - -0.6564
  - -0.1868

- **Zeros**
  - -0.1000

**What is the effect of the zero on the dynamic behaviour?**
Zeros

- Poles and zeros:

  Slow (dominant) Stable zero
Zeros

- Unit step response:

\[
\frac{10s + 1}{s^3 + 9s^2 + 7s + 1}
\]

\[
\frac{\frac{1}{10}s + 1}{s^3 + 9s^2 + 7s + 1}
\]

- Slow dominant zero yields an overshoot.
Example

\[ G(s) = \frac{-\frac{1}{10}s+1}{s^3+9s^2+7s+1} \]

- Poles
  - -8.1569
  - -0.6564
  - -0.1868

- Zeros
  - 10.000

What is the effect of the zero on the dynamic behaviour?
Poles and Zeros

- Dominant Pole
- Fast Unstable Zero
Zeros

Unit Step Response:

➤ Effect of unstable zero is negligible
Example

\[ G(s) = \frac{-10s + 1}{s^3 + 9s^2 + 7s + 1} \]

- Poles
  - -8.1569
  - -0.6564
  - -0.1868

- Zeros
  - 0.1000

What is the effect of the zero on the dynamic behaviour?
Zeros

- Poles and zeros:

![Pole-Zero Map](image)

Slow Unstable Zero
Unit step response:

- Slow unstable zero causes undershoot (inverse response)
Zeros

Observations:

- Adding a stable zero to an overdamped system yields overshoot.
- Adding an unstable zero to an overdamped system yields undershoot (inverse response).

*Inverse response is observed when the zeros lie in right half complex plane, \( \text{Re}(z_i) > 0 \)*

- Overshoot or undershoot are observed when the zero is dominant (closer to the imaginary axis than dominant poles).
Zeros

- Example: System with complex zeros

![Pole-Zero Map](image)

- Dominant Pole
- Slow Stable Zeros
Poles and zeros

- Poles have negative real parts: The system is stable
- Dominant poles are real: yields an overdamped behaviour
- A pair of slow complex stable poles: yields overshoot
Zeros

- Unit step response:

  ➤ Effect of slow zero is significant and yields oscillatory behaviour
Example: System with zero at the origin
Zeros

- Unit step response

- Zero at the origin: eliminates the effect of the step
Observations:

- Complex (stable/unstable) zero that is dominant yields an (overshoot/undershoot)

- Complex slow zero can introduce oscillatory behaviour

- Zero at the origin eliminates (or zeroes out) the system response
Delay

Time required for the fluid to reach the valve usually approximated as dead time

Manipulation of valve does not lead to immediate change in level
Delayed transfer functions

\[ Y(s) = e^{-\theta s} G(s) U(s) \]

e.g. First order plus dead-time

\[ G(s) = \frac{e^{-\theta s} K}{\tau s + 1} \]

Second order plus dead-time

\[ G(s) = \frac{e^{-\theta s} K}{\tau^2 s^2 + 2\xi \tau s + 1} \]
Dead time (delay)

\[ G(s) = e^{-\theta s} \]

- Most processes will display some type of lag time
- Dead time is the moment that lapses between input changes and process response
Delay

- Delayed step response:

![Graph showing a delayed step response with annotations for \( \theta \) and \( \tau \).](image)
Delay

- **Problem**
  - use of the dead time approximation makes analysis (poles and zeros) more difficult

\[
G(s) = \frac{e^{-\theta s} K}{\tau s + 1}
\]

- **Approximate dead-time by a rational (polynomial) function**
  - Most common is Pade approximation

\[
e^{-\theta s} \approx \left(\frac{-\theta s + 2k}{\theta s + 2k}\right)^k, \quad k \in \{1, 2, \ldots\}
\]
In general Pade approximations do not approximate dead-time very well.

Pade approximations are better when one approximates a first order plus dead time process:

\[ G(s) = \frac{e^{-\theta s} K}{\tau s + 1} \approx \left( \frac{2k - \theta s}{2k + \theta s} \right)^k \frac{K}{\tau s + 1} \]

Pade approximations introduce inverse response (right half plane zeros) in the transfer function.
Pade Approximations

- Step response of Pade Approximation

- Pade approximation introduces inverse response