Instructions:

1. This is a 50 minute examination.

2. Please write your student number instead of your name on all answer booklets and sheets submitted.

3. Please state all assumptions made.

4. Calculators are permitted.

5. This is an open-book examination. Textbooks, course notes, assignment and midterm solutions are allowed.

6. Please answer all questions.

7. There are three (3) pages in the question booklet.

8. Good luck!

Please note: “If the instructor is unavailable in the examination room and if doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper a clear statement of any assumptions made.”
Question 1

Figure 1 provides the schematic description of an isothermal continuously stirred tank reactor with a recycle stream. The reactant stream enters the reactor at a flow rate $F$ (m$^3$/s) and molar concentration $C_{A_{in}}$ (mol/m$^3$). The reactor is operated at constant volume $V_1$ (m$^3$). The concentration of reactant $A$ in the tank $C_{A_1}$ (mol/m$^3$) is to be controlled. The product stream enters a separator that recycles the unreacted $A$ to the reactor. The recycle stream re-enters the reactor at a flowrate $F_r$ (m$^3$/s) with a concentration $(1 - \alpha)C_{A_1}$ (mol/m$^3$) where $\alpha \in [0, 1]$ is a small positive number. The final product stream leaves the separator at a flowrate $F_o$ (m$^3$/s) and concentration $\alpha C_{A_1}$ (mol/m$^3$). It is assumed that a first order reaction,

$$A \rightarrow B$$

takes place to produce a chemical $B$ and the rate of reaction of $A$ is given by

$$r_A = -kC_{A_1} \text{ (mol/(m}^3 \cdot \text{s})}$$

where $k$ (s$^{-1}$) is a positive kinetic rate constant. It is assumed that all streams have the same density $\rho$ and that flow is incompressible. One can also assume that the separator holds a negligible volume.

For this process, you are asked to perform the following:
1. Identify the inputs and outputs of the system. Classify inputs as disturbances or manipulated variables.

The inputs are the three flows $F, F_r$ and $F_o$ and the inlet concentration $C_{A_{in}}$. All inputs are either manipulated variables or disturbances depending on the actuators that are available. The outputs are the tank volume $V_1$ and the concentration of $A$ in the tank. (4 marks)

2. Develop a dynamic model that describes the relationship between the inputs and outputs.

We write a mass balance for the tank. The total mass in the system is $\rho V_1$. The rate of mass into the tank is

$$\rho F + \rho F_r$$

and the rate of mass out is

$$\rho F_r + \rho F_o$$

. The mass balance is given by

$$\frac{d\rho V_1}{dt} = \rho F + \rho F_r - \rho F_r - \rho F_o = \rho F - \rho F_o$$

. By the constant volume assumption, we get that

$$F = F_o.$$  

(3 marks)

Next, we write a dynamic molar balance for component $A$. The total moles of $A$ in the tank is equal to $V_1 C_{A_1}$. The molar flow into the system is

$$F C_{A_{in}} + F_r (1 - \alpha) C_{A_1}.$$  

The molar flow out is

$$F_o \alpha C_{A_1} + F_r (1 - \alpha) C_{A_1}.$$  

The rate of reaction of $A$ is

$$-k C_{A_1} V_1.$$  

(3 marks)

The molar balance equation is therefore

$$\frac{dC_{A_1} V_1}{dt} = F C_{A_{in}} + F_r (1 - \alpha) C_{A_1} - F_o \alpha C_{A_1} - F_r (1 - \alpha) C_{A_1} - k C_{A_1} V_1$$

$$V_1 \frac{dC_{A_1}}{dt} = F C_{A_{in}} - F_o \alpha C_{A_1} - k C_{A_1} V_1.$$  

Dividing both sides by $V_1$, we obtain the final equation

$$\frac{dC_{A_1}}{dt} = \frac{F}{V_1} (C_{A_{in}} - \alpha C_{A_1}) - k C_{A_1}.$$  

(12 marks)
3. For your specific choice of inputs and outputs, is the dynamical control system linear or nonlinear?

The most likely scenario in this case would require \( C_{A_{in}} \) to be used as an input to control the concentration in the tank. If one views \( F \) as a disturbances (but such that \( F_o = F \)) then the system is nonlinear in \( C_{A_{in}}, F \) and \( C_{A_{1}} \). If \( F \) is constant, then the system is linear with respect to \( C_{A_{in}} \) and \( C_{A_{1}} \).

(3 marks)

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**Question 2**

Consider the following Laplace domain functions, compute the corresponding time-domain function. Where applicable, use the final value theorem to compute the limit \( \lim_{t \to \infty} y(t) \).

1. (10 marks total) \( Y(s) = \frac{20}{s(s+1)(s^2+9s+20)} \)

We apply partial fraction expansion. We first need to factor the second order term

\[
s^2 + 9s + 20 = (s + 4)(s + 5).
\]

By partial fraction, we get

\[
Y(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s + 1} + \frac{\alpha_3}{s + 4} + \frac{\alpha_4}{s + 5}.
\]

(2 marks) Applying Heaviside expansion,

\[
\alpha_1 = sY(s)|_{s=0} = 1, \quad \alpha_2 = (s + 1)Y(s)|_{s=-1} = -\frac{5}{3},
\]

\[
\alpha_3 = (s + 4)Y(s)|_{s=-4} = \frac{5}{3}, \quad \alpha_4 = (s + 5)Y(s)|_{s=-5} = 20/(-5(-5 + 1)(-5 + 4)) = -1.
\]

(4 marks, 1 mark for each \( \alpha_i \)) Therefore,

\[
Y(s) = \frac{1}{s} - \frac{5}{3(s + 1)} + \frac{5}{3(s + 4)} - \frac{1}{s + 5}.
\]

(1/2 mark) Taking the inverse Laplace transform,

\[
y(t) = 1 - \frac{5}{3}e^{-t} + \frac{5}{3}e^{-4t} - e^{-5t}.
\]

(1/2 mark) The limit as \( t \to \infty \) exists and is equal to 1. (3 marks)

2. (10 marks total) \( Y(s) = \frac{5}{s(s^2+16)} \)

By partial fraction expansion,

\[
Y(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s + 4j} + \frac{\alpha_3}{s - 4j}.
\]
(2 marks) where
\[ \alpha_1 = sY(s)|_{s=0} = \frac{5}{16}, \quad \alpha_2 = (s + 4j)Y(s)|_{s=-4j} = \frac{5}{-4j(-8j)} = -\frac{5}{32}, \]
\[ \alpha_3 = (s - 4j)Y(s)|_{s=4j} = \frac{5}{4j(8j)} = -\frac{5}{32}. \]
(3 marks, 1 mark for each \( \alpha_i \)) Therefore,
\[ Y(s) = \frac{5}{16} - \frac{5}{32(s+4j)} - \frac{5}{32(s-4j)} = \frac{5}{16} - \frac{5}{32} \left( \frac{2s}{s^2 + 16} \right). \]
(1/2 mark) Taking the inverse Laplace transform, we obtain:
\[ y(t) = \frac{5}{16} - \frac{5}{16} \cos(4t). \]
(1/2 mark) In this case, the limit of \( y(t) \) does not exist. Hence, the final value theorem does not apply. (4 marks)

Question 3

Using Laplace transforms, solve the following second order ordinary differential equations:

1. (7 marks total)
\[ \frac{d^2 y(t)}{dt^2} + 6 \frac{dy(t)}{dt} + 8y(t) = 3, \quad \frac{dy(0)}{dt} = 0, \quad y(0) = 0. \]

Taking the Laplace transform on both sides we obtain:
\[ s^2 Y(s) + 6sY(s) + 8Y(s) = \frac{3}{s}. \]
Solving for \( Y(s) \),
\[ Y(s) = \frac{3}{s(s^2 + 6s + 8)} = \frac{3}{s(s+2)(s+4)} \]
(1 mark), where the last equality is obtained by factoring out the quadratic term \((s^2 + 6s + 8) = (s+2)(s+4)\). By partial fraction, we get
\[ Y(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2} + \frac{\alpha_3}{s+4} \]
(1 mark), where
\[ \alpha_1 = sY(s)|_{s=0} = \frac{3}{8}, \quad \alpha_2 = (s + 2)Y(s)|_{s=-2} = -\frac{3}{4}, \]
\[ \alpha_3 = (s + 4)Y(s)|_{s=-4} = \frac{3}{8}. \]
The solution of the ODE is obtained taking the inverse Laplace transform to yield:

\[ y(t) = 3\frac{3}{8} - 3\frac{3}{4}e^{-2t} + 3\frac{3}{8}e^{-4t}. \]

(2 marks)

2. (8 marks total)

\[ \frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = -1, \quad \frac{dy(0)}{dt} = 0, \quad y(0) = 0. \]

Taking the Laplace transform on both sides we obtain:

\[ s^2Y(s) + 4sY(s) + 4Y(s) = -\frac{1}{s}. \]

Solving for \( Y(s) \),

\[ Y(s) = \frac{-1}{s(s^2 + 4s + 4)} = \frac{-1}{s(s + 2)^2} \]

(1 mark) where the last equality is obtained by factoring out the quadratic term \((s^2+4s+4) = (s + 2)^2\). By partial fraction, we get

\[ Y(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{(s + 2)^2} + \frac{\alpha_3}{s + 2} \]

(1 mark), where

\[ \alpha_1 = sY(s)\big|_{s=0} = -\frac{1}{4}, \quad \alpha_2 = (s + 2)^2Y(s)\big|_{s=-2} = \frac{1}{2}, \]

\[ \alpha_3 = \frac{d}{ds}(s + 2)^2Y(s)\bigg|_{s=-2} = \frac{1}{4}. \]

(4 marks, 1 for each \( \alpha_1, \alpha_2 \) and 2 for \( \alpha_3 \)) The solution of the ODE is obtained taking the inverse Laplace transform to yield:

\[ y(t) = -\frac{1}{4} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}. \]

(2 marks)