Block Diagram Manipulation [Section 3.2.]

We often represent control systems using block diagrams. A block diagram consists of blocks that represent transfer functions of the different variables of interest.

If a block diagram has many blocks, not all of which are in cascade, then it is useful to have rules for rearranging the diagram such that you end up with only one block.

For example, we would want to transform the following diagram

![Diagram 1](image1.png)

into

![Diagram 2](image2.png)

How do we get $H(s)$ from $H_1(s), H_2(s), H_3(s), H_4(s)$?

Manipulating and Reducing Block Diagrams [Section 3.2.1]

Since each transfer function represents a linear system, their product is commutative, i.e., for the diagram below

![Diagram 3](image3.png)

$Y(s) = H_2(s) U(s)$ and $U(s) = H_1(s) R(s)$

leads to

$Y(s) = H_2(s) H_1(s) R(s)$
so that the above block diagram can be redrawn as

Now, let’s consider a simple feedback loop:

If we write equations for the above diagram we get

\[ E(s) = R(s) - B(s) \]  \hspace{1cm} (1)

\[ B(s) = H(s) \cdot Y(s) \]  \hspace{1cm} (2)

\[ Y(s) = G(s) \cdot E(s) \]  \hspace{1cm} (3)

Substitute (2) into (1) to get

\[ E(s) = R(s) - H(s) Y(s) \]  \hspace{1cm} (4)

Substitute (4) into (3) to get

\[ Y(s) = G(s) \left( R(s) - H(s) Y(s) \right) \]

i.e., \[ Y(s) \frac{1 + G(s)}{1 + G(s) H(s)} = G(s) R(s) \]

i.e., \[ \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)} \]

The transfer function \( \frac{G}{1 + GH} \) is called the closed-loop transfer function. From the above equation, we can see that the feedback loop can be redrawn as

We have just shown two cases (cascade and feedback) of block diagram reduction. These and other transformations are given in Table 1.
**Table 1. Block Diagram Transformations [Taken from Dorf & Bishop Textbook]**

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Original Diagram</th>
<th>Equivalent Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Combining blocks in cascade</td>
<td>[ X_1 \rightarrow G_1(s) \rightarrow X_2 \rightarrow G_2(s) \rightarrow X_3 ]</td>
<td>[ X_1 \rightarrow G_1G_2 \rightarrow X_3 ] or [ X_1 \rightarrow G_3G_1 \rightarrow X_3 ]</td>
</tr>
<tr>
<td>2. Moving a summing point behind a block</td>
<td>[ X_1 \rightarrow + \rightarrow G \rightarrow + \rightarrow X_3 \rightarrow X_2 ]</td>
<td>[ X_1 \rightarrow G \rightarrow + \rightarrow X_3 \rightarrow X_2 ]</td>
</tr>
<tr>
<td>3. Moving a pickoff point ahead of a block</td>
<td>[ X_1 \rightarrow G \rightarrow X_2 \rightarrow X_3 \rightarrow ]</td>
<td>[ X_1 \rightarrow G \rightarrow X_2 \rightarrow ]</td>
</tr>
<tr>
<td>4. Moving a pickoff point behind a block</td>
<td>[ X_1 \rightarrow G \rightarrow X_2 \rightarrow X_3 \rightarrow ]</td>
<td>[ X_1 \rightarrow G \rightarrow X_2 \rightarrow ]</td>
</tr>
<tr>
<td>5. Moving a summing point ahead of a block</td>
<td>[ X_1 \rightarrow G \rightarrow + \rightarrow X_3 \rightarrow X_2 \rightarrow ]</td>
<td>[ X_1 \rightarrow G \rightarrow + \rightarrow X_3 \rightarrow X_2 \rightarrow ]</td>
</tr>
<tr>
<td>6. Eliminating a feedback loop</td>
<td>[ X_1 \rightarrow + \rightarrow G \rightarrow + \rightarrow X_2 \rightarrow H \rightarrow X_2 \rightarrow ]</td>
<td>[ X_1 \rightarrow \frac{G}{1 \pm GH} \rightarrow X_2 \rightarrow ]</td>
</tr>
</tbody>
</table>
Example [Using the equivalence transformations of Table 1]

Consider the following feedback control system:

First, let's move $H_2$ behind block $G_4$ so that we can isolate the $G_3 - G_4 - H_1$ feedback loop. Use Rule 4 of Table 1 to get

Now, we can eliminate the $G_3 - G_4 - H_1$ loop by using Rule 6 of Table 1:
Now, eliminate the inner loop using Rule 6 of Table 1 again:

\[
\frac{G_2 G_3 G_4}{1 - G_3 G_4 H_1} + \frac{G_2 G_3 G_4 H_2}{1 - G_3 G_4 H_1 \frac{H_2}{G_4}} = \frac{G_2 G_3 G_4}{1 - G_3 G_4 H_1 + G_2 G_3 H_2}
\]

Therefore, the equivalent diagram is

\[
R \rightarrow G_1 \rightarrow \frac{G_2 \ G_3 \ G_4}{1 - G_3 G_4 H_1 + G_2 G_3 H_2} \rightarrow Y
\]

Finally, we can eliminate the last feedback loop to get

\[
R \rightarrow \frac{G_1 \ G_2 \ G_3 \ G_4}{1 - G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4 H_3} \rightarrow Y
\]

Using block diagrams, it becomes easy to see where new blocks can be added to an existing system to alter system performance.

**Note:** Signal-flow graphs are an alternate representation of control systems
- Advantage: no need to do iterative and tedious block diagram manipulations
- See Section 3.2.2 of textbook
Alternate Method for Getting Transfer Functions of Multiloop Systems (without having to do block diagram manipulation)

Suppose we want the transfer function from R to Y of the following multi-component system:

Do the following:

1) Label the outputs of the summing junctions, say \( X_i \)
2) Label the inputs to the summing junctions in terms of \( X_i \) and output Y
3) Write equations at the summing junctions and at the output
4) Eliminate the \( X_i \)'s

So, for the above example, step 1 leads to:
Step 2 leads to:

$$
\begin{align*}
R & + X_1 \\
& \rightarrow G_1 \quad G_1X_1 \\
& + X_2 \\
& \rightarrow G_2 \quad G_2H_2X_2 \\
& + G_2H_1X_2 \\
& \rightarrow H_1 \\
& \rightarrow H_2 \\
& \rightarrow H_3 \\
& \rightarrow Y
\end{align*}
$$

Step 3 gives:

$$
\begin{bmatrix}
1 & G_2 H_1 & 0 \\
-G_1 & 1 + G_2 H_2 & -H_3 \\
0 & -G_2 G_3 & 1
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
Y
\end{bmatrix}
= 
\begin{bmatrix}
R \\
0 \\
0
\end{bmatrix}
$$

Then, using Cramer’s Rule we can do step 4:

$$
Y = \frac{1}{
\begin{bmatrix}
1 & G_2 H_1 & R \\
-G_1 & 1 + G_2 H_2 & 0 \\
0 & -G_2 G_3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & G_2 H_1 & 1 \\
-G_1 & 1 + G_2 H_2 & 0 \\
0 & -G_2 G_3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & G_2 H_1 & 0 \\
-G_1 & 1 + G_2 H_2 & -H_3 \\
0 & -G_2 G_3 & 1
\end{bmatrix}
$$
Cramer’s Rule

If $Ax = B$ is a system of $n$ linear equations in $n$ unknowns such that $\det A \neq 0$, then

$$x_1 = \frac{\det (A_1)}{\det A}, \quad x_2 = \frac{\det (A_2)}{\det A}, \quad \ldots \ldots , \quad x_n = \frac{\det (A_n)}{\det A}$$

Where $A_j$ is the matrix obtained by replacing the entries in the $j^{th}$ column of $A$ by the entries in the matrix

$$B = \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$