Nyquist Plot

Plot of $G(j\omega)$ in the complex plane as $\omega$ is varied on $(-\infty, \infty)$

$s = j\omega$

Nyquist path

Relation to Bode plot

- AR is distance of $G(j\omega)$ from the origin
- Phase angle, $\phi$, is the angle from the Real positive axis
Nyquist Plot

- First order system

\[ K = 1, \tau = 1 \]

\[ AR(0) = 1 \]

\[ AR(-\infty) = AR(\infty) = 0 \quad \phi(\infty) = -\frac{\pi}{2} \]
Nyquist plot

- Second order process: \( G(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} \)

\[ K = 1, \tau = 1 \]

Nyquist Diagram

- \( \xi = 0.25 \)
- \( \xi = 1 \)
- \( \xi = 2 \)

\[ AR(-\infty) = AR(\infty) = 0 \]
\[ \phi(\infty) = -\pi \]
Nyquist plot

- Third order process: \( G(s) = \frac{1}{s^3 + 3s^2 + 3s + 1} \)

\[ AR(-\infty) = AR(\infty) = 0 \quad \phi(\infty) = -\frac{3\pi}{2} \]
Nyquist Plot

- Delayed system

\[ G(s) = \frac{Ke^{-\theta s}}{\tau s + 1} \]

\[ AR(-\infty) = AR(\infty) = 0 \]

\[ \lim_{\omega \to \infty} \phi(\omega) = -\infty \]
Nyquist plot

System with pole at zero

- Pole at $s = 0$ is on the Nyquist path - no finite dc gain
  
  $$AR \to \infty$$

- Strategy is to go around the singularity about $s = \epsilon e^{j\omega}$ in the right half plane ($\epsilon$ a small number)
Nyquist plot

- Pole at zero

\[ s = \epsilon e^{j\omega} \]

\[ s = j\omega \]
System with integral action:

\[ G(s) = \frac{1}{s(s+1)} \]
Che 446: Process Dynamics and Control

Frequency Domain
Controller Design
$G(s) = K_c \left( \frac{1}{\tau_I s} + 1 \right)$

$AR = K_c \sqrt{\frac{1}{\omega^2 \tau_I^2}} + 1$, $\phi = -\tan^{-1}\left( \frac{1}{\omega \tau_I} \right)$
PD Controller

\[ G(s) = K_c \left(1 + \tau_D s\right) \]

\[ AR = K_c \sqrt{1 + \omega^2 \tau_D^2}, \quad \phi = \tan^{-1}(\omega \tau_D) \]
PID Controller

\[ G(s) = K_c \left( \frac{1}{\tau_I s} + 1 + \tau_D s \right) \]

\[ AR = K_c \sqrt{\left( \omega \tau_D - \frac{1}{\omega \tau_I} \right)^2 + 1}, \quad \phi = \tan^{-1} \left( \omega \tau_D - \frac{1}{\omega \tau_I} \right) \]
Consider open-loop control system

\[ Y(s) = G(s)C(s)R(s) = G_{OL}(s)R(s) \]

1. Introduce sinusoidal input in setpoint (\( D(s) = 0 \)) and observe sinusoidal output
2. Fix gain such \( AR = 1 \) and input frequency such that \( \phi = -180 \)
3. At same time, connect close the loop and set \( R(s) = 0 \)

Q: What happens if \( AR > 1 \) ?
Bode Stability Criterion

“A closed-loop system is unstable if the frequency of the response of the open-loop $G_{OL}$ has an amplitude ratio greater than one at the critical frequency. Otherwise it is stable. “

Strategy:

1. Solve for $\omega$ in

$$\arg(G_{OL}(j\omega)) = -\pi$$

2. Calculate AR

$$AR = |G_{OL}(i\omega)|$$
Bode Stability Criterion

To check for stability:

1. Compute open-loop transfer function
2. Solve for $\omega$ in $\phi=-\pi$
3. Evaluate $AR$ at $\omega$
4. If $AR>1$ then process is unstable

Find ultimate gain:

1. Compute open-loop transfer function without controller gain
2. Solve for $\omega$ in $\phi=-\pi$
3. Evaluate $AR$ at $\omega$
4. Let $K_{cu} = \frac{1}{AR}$
Bode Criterion

Consider the transfer function and controller

\[ G(s) = \frac{5e^{-0.1s}}{(s + 1)(0.5s + 1)} \]

\[ C(s) = 0.4\left(1 + \frac{1}{0.1s}\right) \]

- Open-loop transfer function

\[ G_{OL}(s) = \frac{5e^{-0.1s}}{(s + 1)(0.5s + 1)} \left(0.4\left(1 + \frac{1}{0.1s}\right)\right) \]

- Amplitude ratio and phase shift

\[ \text{AR} = \frac{5}{\sqrt{1 + \omega^2}} \frac{1}{\sqrt{1 + 0.25\omega^2}} \left(0.4 \sqrt{1 + \frac{1}{0.01\omega^2}}\right) \]

\[ \phi = -0.1\omega - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}\left(\frac{1}{0.1\omega}\right) \]

- At \(\omega=1.4128\), \(\phi=-\pi\), \(\text{AR}=6.746\)
Bode Criterion

\[ 20 \log_{10}(AR) \approx 15, \quad AR \approx 6 \]

\[ \omega \approx 1.4 \]
Closed-loop tuning relation

- With P-only, vary controller gain until system (initially stable) starts to oscillate.
- Frequency of oscillation is $\omega_c$.

- Ultimate gain, $K_u$, is $1/M$ where $M$ is the amplitude of the open-loop system.
- Ultimate Period

$$P_u = \frac{2\pi}{\omega_c}$$

**Ziegler-Nichols Tunings**

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Bode Stability Criterion

- Using Bode plots, easy criterion to verify for closed-loop stability
  - More general than polynomial criterion such as Routh Array, Direct Substitution, Root locus
  - Applies to delay systems without approximation
  - Does not require explicit computation of closed-loop poles
  - Requires that a unique frequency yield phase shift of -180 degrees
  - Requires monotonically decreasing phase shift
Nyquist Stability Criterion

Observation

- Consider the transfer function \( F(s) = (s + a) \)

- Travel on closed path containing \( s = -a \) in the \( s \) domain in the clockwise direction

- In \( F \) domain, travel on closed path encircling the origin in the clockwise direction

- Every encirclement of \(-a\) in \( s \) domain leads to one encirclement of the origin in \( F \) domain
Nyquist Stability Criterion

Observation

- Consider the transfer function
  \[ F(s) = \frac{1}{(s+a)} \]

- Travel on closed path containing \( s = -a \) in the \( s \) domain in the *clockwise direction*

- In \( F \) domain, travel on closed path encircling the origin in the *counter clockwise direction*

- Every encirclement of \(-a\) in \( s \) domain leads to one encirclement or the origin in \( F \) domain
Nyquist Stability Criterion

- For a general transfer function

\[ F(s) = K \frac{\Pi_{i=1}^{n-1}(s+b_i)}{\Pi_{k=1}^{n}(s+a_k)} \]

- For every zero inside the closed path in the \( s \) domain, every clockwise encirclement around the path gives one clockwise encirclement of the origin in the \( F \) domain.

- For every pole inside the closed path in the \( s \) domain, every clockwise encirclement around the path gives one counter clockwise encirclement of the origin in the \( F \) domain.

- The number of clockwise encirclements of the origin in \( F \) domain \( (N) \) is equal to number of zeros \( (Z) \) less the number of poles \( (P) \) inside the closed path in the \( s \) domain.

\[ N = Z - P \]
Nyquist Stability Criterion

To assess closed-loop stability:

- Need to identify unstable poles of the closed-loop system

\[ A(s)L(s) + B(s)P(s) = 0 \]

- That is, the number of poles in the RHP

Consider the transfer function

\[ F(s) = 1 + G(s)C(s) = 1 + \frac{B(s)}{A(s)} \frac{P(s)}{L(s)} = \frac{A(s)L(s) + B(s)P(s)}{A(s)L(s)} \]

- Poles of the closed-loop system are the zeros of \( F(s) \)
Nyquist Stability Criterion

Path of interest in the \( s \) domain must encircle the entire RHP

- Travel around a half semi-circle or radius \( r \) that encircles the entire RHP \((r \to \infty)\)
- For a proper transfer function
  \[
  F(s) = 1 + G(s)C(s) = 1 + G_{OL}(s)
  \]
- Every point on the arc of radius are such that
  \[
  \lim_{s \to \infty} F(s) = 1
  \]
collapse to a single point \( F(s) = 1 \) in \( F \) domain
Nyquist Stability Criterion

Path of interest is the Nyquist path

Consider the Nyquist plot of $F(s)$
Nyquist Stability Criterion

- Compute the poles of $F(s)$
  
  ➤ Assume that it has $P$ unstable poles
  
  ➤ Count the number of clockwise encirclements of the origin of the Nyquist plot of $F(s)$
    ➤ Assume that it makes $N$ clockwise encirclements
  
  ➤ The number of unstable zeros of $F(s)$ = (the number of unstable poles of the closed-loop system) $= Z$

  $$Z = N + P$$

  ➤ Therefore the closed-loop system is unstable if

  $$Z > 0$$
Nyquist Stability Criterion

Consider the open-loop transfer function

\[ G_{OL}(s) = \frac{B(s)}{A(s)} \frac{P(s)}{L(s)} \]

- \( G_{OL}(s) \) has the same poles as \( F(s) \)

- The origin in \( F \) domain corresponds to the point \((-1,0)\) in the \( G_{OL} \) domain

![Diagram showing Nyquist plot with origin at (-1,0)]
Nyquist Stability

In terms of the open-loop transfer function

- The number of poles of $G_{OL}(s)$ gives $P$

- The number of clockwise encirclements of (-1,0) of $G_{OL}(j\omega)$ gives the number of clockwise encirclements of the origin of $F(j\omega)$ i.e., $N$

- The number of unstable poles of the closed-loop system is given by

$$Z = N + P$$
Nyquist Stability Criterion

“If $N$ is the number of times that the Nyquist plot encircles the point $(-1,0)$ in the complex plane in the clockwise direction, and $P$ is the number of open-loop poles of $G_{OL}$ that lie in the right-half plane, then $Z=N+P$ is the number of unstable poles of the closed-loop characteristic equation.”

Strategy

1. Compute the unstables poles of
2. Substitute $s=j\omega$ in $G_{OL}(s)$
3. Plot $G_{OL}(j\omega)$ in the complex plane
4. Count encirclements of $(-1,0)$ in the clockwise direction
Consider the transfer function

\[ G(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)} \]

and the PI controller

\[ C(s) = 0.4 \left( 1 + \frac{1}{0.1s} \right) \]

The open-loop transfer function is

\[ G_{OL}(s) = G(s)C(s) = \frac{2(0.1s+1)e^{-0.1s}}{0.1s(s+1)(0.5s+1)} \]

- No unstable poles \( P = 0 \)
- One pole on the Nyquist path (must go around small circle around the origin in the right half plane)
Nyquist Stability

- Nyquist plot

- There are 2 clockwise encirclements of (-1,0) \( N = 2 \)
Nyquist Stability

- Counting encirclements must account for removal of origin

\[ G(\epsilon) > 0 \]

\[ N = 2 \]

\[ Z > 0 \quad \text{the closed-loop system is unstable} \]
Nyquist Criterion

Consider the transfer function

\[ G'(s) = \frac{(s+3)}{(s+5)(s+7)} \]

and the PI controller

\[ C'(s) = 9\frac{(s+1)}{s} \]

The open-loop transfer function is

\[ G_{OL}(s) = G(s)C'(s) = \frac{9(s+3)(s+1)}{s(s+5)(s+7)} \]

- No unstable poles \( P = 0 \)
- One pole on the Nyquist path (must go around small circle around the origin in the right half plane)
Nyquist Stability

- Nyquist plot

- There are 2 clockwise encirclements of (-1,0) \( N = 0 \)
Nyquist Criterion

Consider the transfer function

\[ G(s) = \frac{4e^{-4s}}{s+7} \]

and the PI controller

\[ C(s) = \frac{7(s+1)}{2s} \]

The open-loop transfer function is

\[ G_{OL}(s) = G(s)C(s) = \frac{28e^{-4s}(s+1)}{2s(s+7)} \]

- No unstable poles \( P = 0 \)
- One pole on the Nyquist path (must go around small circle around the origin in the right half plane)
Nyquist Stability

- Nyquist plot

- There are 2 clockwise encirclements of (-1,0) \( N > 0 \)
Bode Stability

- Stability margins for linear systems

\[ G_{OL}(j\omega) \]

\((-1, 0)\)
Stability Considerations

- Control is about stability

- Considered exponential stability of controlled processes using:
  - Routh criterion
  - Direct Substitution
  - Root Locus
  - Bode Criterion (Restriction on phase angle)
  - Nyquist Criterion

- Nyquist is most general but sometimes difficult to interpret

- Roots, Bode and Nyquist all in MATLAB

\[ \text{Polynomial (no dead-time)} \]
Stability Margins

- Stability margins for linear systems

\[ G_{OL}(j\omega) \]

\((-1, 0)\)

\[ M_g \]

\[ M_f \]
Stability Margins

- **Gain margin**: $M_g$
  
  Let $AR_u$ be the amplitude ratio of $G_{ol}$ at the critical frequency $\omega_u$

  \[ M_g = \frac{1}{AR_u} \]

- **Phase margin**:
  
  Let $\phi$ be the phase shift of $G_{ol}$ at the frequency where $AR = 1$

  \[ M_f = 180 + \phi \]
Stability Margin